CANTOR AND THE BURALI-FORTI PARADOX

Introduction

In studying the early history of mathematical logic and set theory one typically reads that Georg Cantor discovered the so-called Burali-Forti (BF) paradox sometime in 1895, and that he offered his solution to it in his famous 1899 letter to Dedekind. This account, however, leaves it something of a mystery why Cantor never discussed the paradox in his writings. Far from regarding the foundations of set theory to be shaken, he showed no apparent concern over the paradox and its implications whatever.

Against this account, I will argue here that in fact Cantor never saw any paradox at all, but that his conception of set at that time, and already as far back as 1883, was one in which the paradoxes cannot arise.

The main sources I will draw on are Cantor's great 1883 work Grundlagen einer allgemeinen Mannigfaltigkeitslehre and his "Mitteilungen zur Lehre vom Transfiniten." The clue Cantor gives us is his claim that sets are related to the mikta (more commonly, meikta) of the Philebus. Cantor is silent on exactly what the relation is, so we will have to tease it out for ourselves. This will require a bit of a digression.

In the Philebus, a meikton is associated with two other well known Greek concepts, peras and apeiron. These latter two are of Pythagorean origin, and in fact head the so-called Pythagorean "table of opposites." The Pythagoreans, of course, held that things are numbers. For the later

Infinite Sets and Plato's MEIKTA

An important clue for understanding Cantor aright has been hitherto overlooked. In a significant footnote to the Grundlagen Cantor defines a set (Menge) as "any plurality that can be thought of as a unity (jedes Viele, welches sich als Eines denken lässt)". He then adds the following.

I believe that what I've defined here is related to the platonic eidos or idea, as well as to what Plato calls mikton in his dialogue Philebus. He contrasts this with the apeiron, i.e., the unlimited (Unbegrenzten), the indeterminate (Unbestimmten), which I call the nongenuine infinite, and also with the peras, i.e., the limit, and explains it as an ordered "mixture" of the two.
Pythagoreans around the time of Socrates this meant that numbers are the basic cosmological building blocks of which all other things are comprised. This view no doubt arose out of observations of the mathematical relations that obtain in and among objects in the world. It followed for them that if things are numbers in this sense, then the elements, or principles, of number are also the elements of things. Aristotle tells us these elements are Odd and Even, which in turn are identified with peras (the limited, the determinate) and apeiron (the unlimited, the indeterminate).8

In the Philebus Plato’s use of ‘peras’ and ‘apeiron’ reflects their Pythagorean roots. In the well known “gift of the gods” passage he writes that “all things . . . that are ever said to be consist of a one and a many and have in their nature a conjunction of Limit (peras) and Unlimited (apeiron).”9 In a later passage he reiterates this, saying that “God had revealed two constituents of things, the Unlimited and the Limit.”10 It is in this latter passage that ‘meikton’ and its cognates appear. The passage concerns itself with a division of “all that now exists in the universe”11 into four classes. The first consists of all instances of apeiron, i.e., the various apeira in the world, and second all instances of peras, the various perata. The third class consists of meikta, the products that result from the union or mixture of members of the first two classes, and the last contains the causes of the mixtures. I will discuss each class in turn.

The class of apeira, according to Plato, contains phenomena like the hotter, the colder, the drier, the moister,12 i.e., “[e]verything we find that can become more and less, and admits of strength and mildness, too much, and everything of that sort.”13 A particular apeiron then is a phenomenon like heat that admits of degrees. Typically such a phenomenon will be representable as a linear continuum; heat, humidity, pitch, tempo can all take any of a continuum of values. For any particular value there is always another that is greater and another that is smaller. Hence, what ‘heat’, ‘humidity’, etc. refer to are indefinite, indeterminate, apeiron: no one specific thing is picked out. Of heat (hotter and colder), for example, Plato writes: “[t]his pair is always without bounds; and being boundless means, I take it, that they must be absolutely indeterminate (apeiron).”14

The class of perata on the other hand contains “[t]hings that don’t allow of these features [i.e., more and less, strength, mildness, etc.] but admit of all the opposite things—equal and equality, and after equal, double and every proportion of number to number or measure to measure . . . .”15 Thus, this class seems to consist simply of rational numbers, for it is these that represent equality (e.g., 2:2, perhaps also 3:2:6:4), double (4:2), and indeed “every proportion of number to number or measure to measure.”16
character of rational numbers, then, stands in stark contrast to the indeterminacy of the various apeira.

Plato's third class is comprised of meikta, the products that result from the mixture of peras and apeiron. This is clarified most effectively by an example. Pitch and tempo, as Plato points out, are both apeira. Pitch is characterized by a continuum of tones that extends from the lowest sounds perceptible to the human ear to the highest. Similarly, tempo ranges on a continuum from arbitrarily slow (e.g., one beat per century) to arbitrarily fast (e.g., one beat per 1/1000 seconds). Now, it is a familiar fact that the twelve tones in an octave can be represented in terms of simple ratios between natural numbers. Once one has determined a given point on the continuum of pitch to serve as the tonic, one can then specify the location of the other eleven tones in terms of these ratios. The resulting twelve-tone scale is thus a meikton, a product of the correct blend of peras (the ratios that determine the scale) and apeiron (pitch).

The introduction of tempo enables one to construct further, more complex meikta. Given a particular tempo (e.g., one beat per 1/2 second), which is itself a meikton, one can assign to each of a succession of notes a length of time that determines how long it is to be played or sung before moving to its successor. The relation of these lengths of time to the tempo is expressible in terms of rational numbers. The resulting meikton of pitch and tempo is a song. This particular way of blending peras and apeiron, then, "establishes perfectly the whole art of music."

Finally, by the cause of a meikton Plato simply means an intelligence, a rational individual. Thus, in keeping with the above, a musician is the cause of a tuneful meikton. The fourth class is the class of all such causes.

Now, the examples of meikta Plato gives, e.g., musical works, fair weather, a person's health, and so on, don't in any way seem to be pluralities thought of as unities. So whatever connection Cantor had in mind between sets and meikia, it was not that they are similar kinds of things. But what then is the connection?

The answer has to do with Cantor's defense of the actual infinite. According to Cantor, two related ideas lie at the heart of all antiinfinitistic arguments, viz.,

(fin) Number is essentially finite

and

(inf) The infinite is indeterminate.

Historically (fin) is rooted in the idea that a number is something one can build up to by the repeated addition of units; in St. Thomas's words (which
Cantor quotes), “every number is multitude measured by one.” 23 Any number can be seen as the completed, end result of such a building-up process. It follows that every number is essentially finite.

Now for Cantor this argument for (fin), as well as all others, begs the question:

All so-called proofs against the possibility of actual infinite numbers are . . . flawed through and through . . . [in that] at the outset they assume or even force all the properties of finite numbers upon the numbers in question . . . . 24

It is quite true that one can never build up to an infinite number by the repeated addition of units. But all that shows is that infinite numbers cannot be defined in such terms. The truth of the matter, Cantor argues, is that the repeated addition of units is just the first of two “principles of generation (Erzeugungsprinzip)” that can be used for defining new numbers from previously given ones. 25 The first principle gives us the class (I) of all finite numbers. We note however that there is in (I) no number which numbers the entire class the way that a finite number can number a finite set. Rather than simply supposing that (I) is indeterminate, apeiron, as his predecessors had done, Cantor remarks that there is “nothing objectionable in conceiving of a new number” 26 which numbers, or “counts” as it were, the class (I) as it is given, i.e., ordered according to magnitude. This new number Cantor calls ω, and it is “the first whole number following all the [finite] numbers.” 27 It is defined by means of Cantor’s second principle of generation, viz., given a sequence S of previously generated numbers, a further number can be conceived which is the first number greater than, hence following, all the numbers in S. 28 Once ω has been defined, then, we can use the first principle of generation once again to define ω + 1, ω + 2, . . ., ω + n, . . ., and then again, by the second principle, the number ω + ω, then (ω + ω) + 1, (ω + ω) + 2, . . ., (ω + ω) + ω, and so on without limit.

Lest we simply “lose ourselves in the unlimited,” 29 Cantor introduces a third principle, the “principle of limitation (Beschränkungsprinzip),” 30 which allows us to form a well-defined increasing sequence of number classes. Cantor notes that the first two principles alone generate only “countable” numbers that meet the following condition: for any such number, the set of all its predecessors has the power of (or, is countable by) the first number class (I). Cantor thus defines the second number class (II) by “limiting” it to all such numbers. By his second principle of generation, one is then able to conceive of the first number exceeding all the numbers in (II), viz., the first number, call it ω1, such that the power P of the set of its predecessors is greater than the power of (I). P then is the power of (I) ∪ (II), which is just the power of (II). As Cantor shows in the next section of the Grundlagen, P is
in fact the next greatest power after the power of (I). Given \( \omega 1 \), then, we can begin all over again with \( \omega 1 + 1, \omega 1 + 2, \ldots, \omega 1 + n, \ldots, \omega 1 + \omega 1, \ldots \). By the principle of limitation we can now define the number class (III) of all numbers \( \alpha \) such that the power of the set of all \( \alpha \)'s predecessors is P. It is then possible to conceive of the first number, \( \omega 2 \), such that the power of the set of its predecessors is greater than P, from which we can go on to form the next number class (IV), and then (V), and so on without end.

Without the second of the two principles of generation it is easy to understand how the infinite could come to be thought of as indeterminate, as (inf) states. One determines a given multitude by giving its number, in effect, counting it; but if the multitude is infinite, it has no number (by (fin) ) and hence cannot be determined. The most one can say is that no matter what number you try to number the multitude with, the number is always too small; no number built up by the repeated addition of units could be the number of an infinite multitude. Unlike well-behaved, completed, finite sets, then, an infinite set is unruly, unfinished, indeterminate; in short, *apeiron.*

Cantor points out that this line of thought is corroborated by two well known types of infinity, one mathematical, one theological. The first, familiar since Aristotle, is what Cantor calls the potential or nongenuine infinite (*Eineigentlich-unendliche*), by which he means “a variable quantity that either grows beyond all limits or diminishes to arbitrary smallness but always remains finite.”

The paradigm Cantor has in mind here is the sort of infinity occurring in analysis. For example, the infinite sum \( \sum_{i=1}^{\infty} a_i \) is defined as the value which the sequence of its partial sums *approaches* (supposing the sequence converges) as \( i \) grows larger (i.e., “approaches infinity”), but nowhere is it assumed that \( i \) takes anything other than finite, albeit ever-increasing, values. However, the stock of values \( i \) can take is never exhausted; it “grows beyond all limits;” no matter what value \( i \) takes, there is always another beyond it. The process is thus never “completed” or “finished,” but is indeterminate, *apeiron.*

The second sort of indeterminate infinite, one with roots in Aquinas, Locke, Descartes, and especially Spinoza, Cantor calls “the true infinite or absolute.” This, he tells us, is the sort of infinity which is “in God.” Unlike the potential infinite, which signifies only the possibility of boundless finite increase or diminution, the absolute is an unchanging actual infinite. Viewed as the measure of God's nature, it is an “absolute quantitative maximum,” exceeding all determinable magnitudes; it “surpasses, as it were, all human comprehension and in particular evades all mathematical determination.”

Prior to Cantor, then, one had only two options regarding infinite sets: one could reject the possibility of an actually infinite multitude and subscribe
to some form of potential infinite; or one could embrace such a possibility, but, with only the absolute to serve as a model, one would have to acknowledge the multitude's essential incomprehensibility and indeterminateness. Cantor presented a third option. Let us note first that Cantor believed implicitly that every set can be well-ordered.\textsuperscript{40} Every well-ordered set, he tells us in Section 2 of the \textit{Grundlagen}, has what he calls a unique \textit{Anzahl}, or in his later terminology, a unique order type.\textsuperscript{41} Now, \textit{Anzahlen}, Cantor states, can be identified with his newly defined numbers:

\begin{quote}
[T]his concept [i.e., \textit{Anzahl}] is always expressed by means of a determinate number of our extended number field.\textsuperscript{42}
\end{quote}

Thus, since every set is well-orderable, and every well-ordered set has an \textit{Anzahl}, it follows that every set can be numbered. Herein, finally, lies the connection between sets and Plato's \textit{meikta}. \textit{Meikta} are "ordered (geordnete) mixtures" of \textit{apeiron} and \textit{peras}, where the latter, in the \textit{Philebus}, is essentially the concept of (rational) number. Prior to being ordered, an infinite set is in a certain sense \textit{apeiron}; it is capable of being ordered in uncountably many different ways\textsuperscript{43} and hence has no intrinsic \textit{Anzahl}, no intrinsic number.\textsuperscript{44} The unordered set is like a jumble of notes waiting to be ordered into a melody. By ordering an infinite set we bring number, \textit{peras}, to an \textit{apeiron}, as a musician brings number and order to mere notes. The result is a \textit{meikton}, a determinate mixture of \textit{peras} and \textit{apeiron}, number and the indeterminate infinite.

Infinite sets, then, while indeed essentially different from finite sets, nevertheless share with finite sets the property of being determinable by well-defined numbers. Thus, the idea that infinite sets are essentially indeterminate, simply because they are infinite, loses its foundation; in so far as they can be well-ordered, infinite sets are comprehensible and determinate. In Cantor's words:

The assumption that except for the absolute, which cannot be reached by any determination, and the finite there can be no modifications which, although not finite are nevertheless determinable by numbers, and are thus what I call genuinely infinite, I find wholly unjustified.\textsuperscript{45}

Thus, in place of the scholastic proposition "\textit{infinitum actu non datur}," Cantor suggests the proposition

\begin{quote}
Omnia seu finita seu infinita \textit{definita} sunt et excepto Deo ab intellectu determinari possunt. (All things, whether finite or infinite, are \textit{definite} and, except for God, can be determined by the intellect.)\textsuperscript{46}
\end{quote}

So in likening sets to the \textit{meikta} of the \textit{Philebus} Cantor was challenging an ancient, well-entrenched philosophical position which had associated, even
identified, the infinite with the indeterminate. To the contrary, armed with the concept of a well-ordering and his extended number sequence Cantor showed that infinite sets are no less open to mathematical determination than finite sets, and are thus no less legitimate. Doubts about their validity, and that of the concept the infinite in general, are unfounded.

All that is essential in order for the assertion that a given set is actually infinite to be valid is that all the elements of the set be definite, distinct entities and that the Anzahl of the set [for any given well-ordering] be greater than every finite number.

**Paradoxes and the Absolutely Infinite**

As is well known, there are certain collections which, although well-orderable, nonetheless cannot be numerically determined, viz., what we now call proper classes. Such classes are just "too big" to have a number. It follows from the above that they cannot be *meikta*. Cantor, who was fully aware of such collections, in effect says as much in the *Grundlagen*. In constructing the successive transfinite numbers and number classes, he writes,

we will proceed farther and farther, never reaching an insurmountable limit (Grenze) nor even an approximate comprehension of the absolute. The absolute can only be acknowledged, never known, not even approximately. For just as for any finite number in the first number class (I), no matter how large, one always has before one the same power of numbers greater than it, so for any transfinite number of one of the higher number classes, no matter how large, there follows a totality of numbers and number classes which is not reduced in the least in power compared to the whole infinite totality of numbers from 1 on . . . Thus, the absolutely infinite number sequence seems to me in a certain sense to be an appropriate symbol of the absolute . . .

The absolute, we recall, is an "absolute quantitative maximum" that "evades all mathematical determination," i.e., has no number, and is to be strictly distinguished from the numerically determinable transfinite. This is why Cantor found his extended number sequence to be "an appropriate symbol of the absolute;" there is no number which is the number of the sequence of all numbers; the sequence evades all mathematical determination. As such, like the absolute itself, it cannot be known or comprehended. Already in the *Grundlagen*, then, Cantor was aware of the special status of "absolutely infinite" collections and how they differ from other infinite collections, which, as *meikta*, are both mathematically determinable and knowable. Cantor, that is, was aware of the distinction between sets and proper classes, though undoubtedly not yet fully cognizant of its significance.

These considerations have important consequences for our understanding of the relation between Cantor and the set theoretic paradoxes, es-
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Especially the account of Cantor’s alleged discovery of the BF paradox mentioned in the Introduction. For if the above is correct, then it is difficult to believe that Cantor could have seen any paradox at all. To see this, consider the two propositions that generate the paradox:

BF1 The extended number sequence S is a well-ordered collection.

BF2 Every well-ordered collection has a definite Anzahl which is expressed by a unique number in S.

Cantor accepted BF1, but he surely would have rejected BF2 out of hand; as he had already made abundantly clear in the Grundlagen and the “Mitteilungen,” some well-ordered collections, in particular, his extended number sequence S, are absolutely infinite and hence fall beyond the scope of mathematical determination. Thus, Cantor was no more inclined to think that S’s “order type” could be expressed by a unique transfinite number in S than he was to think that the order type of the sequence of finite numbers S’ could be expressed by a unique finite number in S’, an assumption which generates a paradox nearly identical to Burali-Forti’s.

The BF paradox, then (as well as the other set theoretic paradoxes), is based on a failure to distinguish the transfinite from the absolutely infinite, an error against which Cantor had explicitly warned in the “Mitteilungen” in the mid-1880’s. Having made the distinction himself, with explicit reference to S, it follows that Cantor could not have seen any paradox. In this light the rather striking lack of discussion of this and the other paradoxes in Cantor’s writings ceases to be mysterious.

What then are we to make of Cantor’s 1895 “discovery” of the BF paradox and his apparent “solution” to it in his letter to Dedekind? Given the above, I think the answer can be derived from the letter itself. He begins the mathematically relevant portion of the letter as follows:

As you know, many years ago I had already arrived at a well-ordered sequence of cardinalities (Mächtigkeiten) or transfinite cardinal numbers, which I call “alephs.” . . .

The big question was whether, besides the alephs, there were still other cardinalities of sets; for two years now I have been in possession of a proof that there are no others, so that, for example, the arithmetical linear continuum (the totality of all real numbers) has a definite aleph as its cardinal number.

The rest of the letter leads up to the proof of the theorem which answers the “big question,” viz., whether or not

(T) All cardinals are alephs.

The focus of the letter, then, has nothing to do with any paradoxes or inconsistencies in set theory, but with a vitally important unsolved problem. This
problem was especially significant for Cantor, as he himself points out, because of its relevance to the continuum hypothesis, with which he had been obsessed for many years. If there are cardinals that are not alephs, the possibility remains open that the cardinality of the continuum is not an aleph and hence that the continuum hypothesis can never be proved. Thus, showing that all cardinals are alephs constituted a major step towards its solution.

The proof of (T) involves the distinction Cantor had discovered sixteen years before between mathematically indeterminable absolutely infinite classes and mathematically determinable sets, those collections likened to meikta. It must have seemed natural to Cantor to approach (T) with this distinction in mind, for in an absolutely infinite class like S he already had an example of a collection that has no ordinal number and hence no corresponding aleph. The distinction in the letter is presented in language that clearly recalls the definition of 'set' in the Grundlagen, but invokes also the notion of consistency.

If we start from the notion of a definite multiplicity (Vielheit) . . . of things it is necessary, as I discovered, to distinguish two kinds of multiplicities . . . .

For a multiplicity can be such that the assumption that all of its elements 'are together' leads to a contradiction, so that it is impossible to conceive of the multiplicity as a unity (Einheit), as 'a completed thing.' Such multiplicities I call absolutely infinite or inconsistent multiplicities . . . .

If on the other hand the totality of the elements of a multiplicity can be thought of without contradiction as 'being together,' so that they can be gathered together into 'one thing,' I call it a consistent multiplicity or 'set.'

The definition of 'set' or 'consistent multiplicity' here does not differ essentially from the Grundlagen definition. Now however Cantor adds an explicit definition of the concept of an absolutely infinite multiplicity: in contradiction to a set, an absolutely infinite multiplicity cannot be thought of as a unity; the attempt to do so will lead to a logical contradiction. The discussion of the BF paradox above shows that Cantor could hardly have found this surprising; indeed, he might well have expected it, since it results from an attempt to comprehend the absolute (or at least an appropriate symbol thereof), to treat that which 'proceed[es] farther and farther, never reaching an insurmountable limit' as a 'completed thing.'

The motivation behind the explicit characterization of absolutely infinite multiplicities in the 1899 letter seems clear. Cantor's (faulty, as it turns out) proof of (T) shows that if a multiplicity V has no aleph as its cardinal number, then V is absolutely infinite. He shows this with the help of an axiom (actually an early form of the axiom of replacement).

(Ax) Two equivalent multiplicities are either both sets or are both absolutely infinite
and a crucial lemma

(L) The extended ordinal number sequence $S$ is absolutely infinite.

Given its dependence on (Ax) and (L), it is clearly essential for the proof of (T) that the notion of absolute infinity be mathematically tractable; it must at least be clarified to the extent that it can be proved that a given multiplicity like $S$ is absolutely infinite. Notions like "the sort of infinity that is in God" or "proceeding farther and farther without limit" won't do the job. Cantor thus focuses on the feature of mathematical indeterminability: what is mathematically most characteristic about absolutely infinite classes is that they cannot be determined by any number. How then can one most easily prove that a given class is absolutely infinite? Just suppose that, like any set, it can be conceived of as a "completed thing" with a definite number. If you can derive a contradiction you know the class is absolutely infinite; hence the term 'inconsistent multiplicity'.

This is precisely the method Cantor uses to prove (L), and the contradiction he derives to prove it, on the assumption that $S$ is not absolutely infinite, is of course the BF paradox. Since he possessed the distinction between absolutely infinite multiplicities and sets, however, the contradiction was no more a paradox for him than is the contradiction one derives from the assumption that $\sqrt{2}$ is rational for someone who possesses the distinction between rational and irrational numbers.

So the idea that Cantor first discovered and then, in rather ad hoc fashion, solved the BF paradox (as well as the "largest cardinal" paradox) is incorrect. The paradoxes can only breed in an environment that does not recognize the distinction between consistent and inconsistent multiplicities, sets and proper classes. But this is a distinction Cantor made long before his 1895 "discovery." The discovery itself was nothing more than a reductio proof of a proposition he had believed all along, and which required a more rigorous foundation in order to be used to demonstrate that all cardinals are alephs.

**APPENDIX: TEXTUAL INCONGRUITIES**

I believe that in light of the textual evidence the claims above are scarcely deniable. Nonetheless, there a couple of textual incongruities that deserve some discussion.

(1) In the *Grundlagen*, in a passage in fact quoted in the previous section, Cantor speaks of the "power" of the extended number sequence $S$. But powers, it might be objected, are just cardinal numbers, so Cantor must have thought $S$ had a definite cardinality, thus making his theory susceptible to paradox.
This cannot be right. As I've argued above, there is little doubt that Cantor held that \( S \), as an appropriate symbol of the absolute, is mathematically indeterminable by any ordinal number. However, in the *Grundlagen* Cantor explicitly states (without sufficient justification, as he came to see) that every power \( P \) is the power of some number class \( (N) \) and that for any power \( P \) and set \( M \), if the power of \( M \) is \( P \), \( M \) is mathematically determinable ("countable," as Cantor puts it) by a number of the \( N + 1 \)st number class.\(^6\) It is difficult to believe Cantor could have assented to both of these propositions as he did and at the same time could have ascribed a definite power to the mathematically indeterminable sequence \( S \), since this would immediately imply that it is not mathematically indeterminate.

The solution I think is just that, in the passage in question, Cantor was thinking of 'power' in the broader sense in which he had first defined it in 1878, a sense which applies to both sets and proper classes.\(^6\) In this sense a power is a property of equivalent aggregates, a sort of intensional Fregean view.

If two well-defined aggregates (Mannigfaltigkeiten) \( M \) and \( N \) can be put into one to one correspondence with each other (sich eindeutig und vollständig, Element für Element, einander zuorden lassen) \ldots \), then in what follows I will say that they have the same power, or also, that they are equivalent.\(^6\)

On this construal, it makes perfect sense to talk about \( S \)'s power. But this power is an "absolute quantitative maximum" subject to no increase\(^6\) and hence differs essentially from the transfinite powers, of which there is no greatest (which Cantor also proved in the *Grundlagen*).

(2) Rather more serious is the fact that Cantor without qualification twice calls the class \( C \) of all cardinal numbers, which he clearly saw to be absolutely infinite,\(^6\) a set (Menge). Neither of these occurrences is in the *Grundlagen*, where the only actual definition of the concept of set appears prior to the letter to Dedekind. The first is in a footnote in the "Mitteilungen." After noting that there are "uncountably many ways of collecting and ordering the finite cardinal numbers," he writes:

Later we will see that the totality (Gesamtheit) of all cardinal numbers or powers \ldots \, when conceived of as ordered according to magnitude, forms a well-ordered set (Menge).\(^6\)

Two considerations are relevant here. First, the distinction between sets and absolutely infinite classes, though well known to Cantor, as we've seen, was not yet of vital significance at the time of the publication of the "Mitteilungen." Cantor was not yet worried about whether there were any powers that were not determined by any number class, i.e., whether there were any cardinal numbers that were not alephs, nor had any of the "paradoxes" ap-
peared, for which the distinction between sets and absolutely infinite classes provided the necessary corrective. Second, absolutely infinite classes often behave just like sets. In particular, they can often be well-ordered, as in the passage quoted above; the class $C$ of cardinals can be well-ordered according to magnitude. That it can be is Cantor’s point in the passage; the emphasis is not on $C$’s being a set but on its being well-orderable. Given both considerations, it is thus not surprising that we should find a place or two where Cantor’s terminology is loose. This is nowhere better illustrated than at the second place where Cantor speaks of $C$ as a set. The passage is in his 1891 article which proves that in general for any set $M$, the power set of $M$ has a greater cardinality than $M$, from which it follows that $C$ has no maximum.\(^69\)

I have already shown in the Grundlagen... by entirely different means that the powers have no maximum. It was proved there that the totality (Inbegriff) of powers, when thought of as ordered according to magnitude, forms a “well-ordered set (Menge)” such that... for every power there is a next greater, and also for every set (Menge) of powers that increases without end there follows a next greater power.\(^70\)

Clearly Cantor cannot be using ‘set’ univocally in this passage. If he were, then since $C$, ordered according to magnitude, is a “set of powers that increases without end,” it would follow that there is a power greater than all the powers in $C$, which, aside from being contradictory in itself, also contradicts what Cantor had just asserted, “that the powers have no maximum.” What Cantor has in mind by sets of endlessly increasing powers, of course, are bounded subsets of $C$ like, for example, $|\aleph_0, \aleph_1, \ldots, \aleph_n, \ldots|$, which has $\aleph_\omega$ as its least upper bound, i.e., as the “next greater power.” So in calling $C$ a set here Cantor is plainly using the term in an extended sense. And in fact this is precisely how he puts it four years later in the first (1895) installment of his “Beiträge zur Begründung der transfinite Mengenlehre.”\(^71\)

It will be shown that the transfinite cardinal numbers can be ordered according to magnitude and that so ordered, like the finite cardinals, they form a “well-ordered set,” although in an extended (erweiterten) sense.\(^72\)

Note that what Cantor needs in order to prove this is just proposition (T), that all cardinals are alephs. He did not in fact prove (T) to his satisfaction in time to make good this promise in the second (1897) installment of the “Beiträge.” As argued above, the reason Cantor began looking more deeply into absolutely infinite classes was to prove (T), which in turn led him to see the necessity of rigorously distinguishing them from mathematically determinable sets. It seems to me then that Cantor’s qualification of ‘well-ordered set’ in this passage stems from his incipient attempts to prove (T), and not
from "skeptical doubts about the concept of all ordinal or cardinal numbers," as Zermelo claims in his notes on the "Beiträge." 7 To the contrary, Cantor never had such doubts.

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NOTES


2. After completing the present paper I discovered a fascinating and informative article by G. H. Moore and A. Garciadiego in which it is pointed out that Cantor did not see a paradox. The present paper shows why. See "Burali-Forti's Paradox: A Reappraisal of its Origins," Historia Mathematica, 8 (1981): 319-50.

3. Ibid., 165-209.

4. GA, pp. 378-439. The "Mitteilungen" consist of a series of revealing letters with added footnotes written between February 1884 and May 1886 to a number of mathematicians and theologians who had questioned Cantor on various points of the Grundlagen.

5. GA, p. 204, n1. All translations from GA are my own unless otherwise indicated.

6. Ibid. I've substituted transliterations of the Greek terms in this passage.


9. Philebus, 15c-d.

10. Ibid., 23c.

11. Ibid., 23c-d.

12. Or: hotter-colder, drier-moister, as it is sometimes expressed.


15. Ibid., 25a-b.

16. The extent of this class (e.g., whether it includes all and only rational numbers) is not clear from the text. Gosling holds that there are important restrictions on exactly what the class contains, viz., only those ratios that are specified by a given techné like music or medicine. How the issue is ultimately resolved, however, won't affect matters here. Cf. Plato, Philebus, tr. and comm. by J. C. B. Gosling (Oxford: Clarendon Press, 1975) p. 92.

18. Hackforth claims that any given note, regardless of its relation to other notes, is also a *meikton* in itself. He is speaking about temperature in the passage, but his remarks apply to any continuum. Cf. R. Hackforth (tr.), *Plato’s Philebus* (Cambridge: Cambridge University Press, 1972), p. 42.

19. Since it arises by specifying a particular ratio of beats to an interval of time.

20. As our musical terminology today bears witness, e.g., half notes, quarter notes, eighth notes, etc.


22. Dauben incorrectly remarks that “meikton” refers to the same thing as “edos” and “idea”, viz., presumably, Plato’s Ideas (*GC*, p. 170). To the contrary, there is some dispute as to whether the Ideas appear in the *Philebus* at all; and in any case it is certain they cannot be identified with *meikta*. Cf., e.g., Hackforth, *Plato’s Philebus*, p. 39.

23. *Summa Theologiae*, 1, qu. 7, art. 4; *GA*, p. 403.


26. Ibid.

27. Ibid.


30. *GA*, p. 197. Cantor also calls this the *Hemmungsprinzip*.

31. By means of a number of long footnotes and scattered references Cantor shows that this association of the infinite with the indeterminate developed into an established tradition extending right up to his own time. Origen, for instance, claimed that, due to its irrational, indeterminate character, “by its nature whatever is infinite is incomprehensible.” (Origen, *De Principiis*, Bk. 2, ch. 9, sec. 1; *GA*, p. 403.) In fact, as Cantor also points out, Origen went so far as to claim that “if the divine power were infinite, of necessity it could not even understand itself, since the infinite is by its nature incomprehensible.” St. Thomas argued that there could not be a created infinite multitude, because God could not have had a “definite intention” to create one. As I understand it, the argument runs as follows: Everything created has to be “comprehended under some definite intention of the Creator.” For a created multitude this means being “comprehended under a certain number.” But the infinite, being indefinite, has no number. So God could not have had a definite intention to create an infinite multitude. (*Summa Theologiae*, 1, qu. 7, art. 4; *GA*, pp. 403–04.) In the modern period Descartes proclaimed “that all that in which we find no limits is indefinite, . . . ” and also, echoing Origen, that “the idea of the infinite, in order to be true, cannot by any means be comprehended, since this very incomprehensibility is comprised within the formal concept of the infinite.” (E. H. Haldane and G. R. T. Ross (tr.), *The Philosophical Works of Descartes* (Cambridge: Cambridge University Press, 1972) vol. 1, p. 229, vol. 2, p. 218. Cf. *GA*, p. 175.) Finally, Cantor’s contemporary J. F. Herbart claimed that it is impossible to count an infinite set, not because of our finite limitations but “because the true infinite can only be conceived as an indeterminate, uncompleted thing.” (*GA*, p. 392, fn.)
32. Occasionally he also calls it the "syncategorematic" infinite; cf. GA, pp. 206, 373, 404.

33. GA, p. 165.

34. As Cantor himself points out, though, by means of a clever metaphor, while the limit process in analysis does not presuppose that the variable which takes the successively increasing or diminishing values take any actually infinite (or infinitesimal) value, it does seem to presuppose that the entire set of values the variable takes forms an actually infinite collection. GA, pp. 392–93. Cf. also Dauben, GC, p. 127.

35. Cf. Aquinas, Summa Theologiae, I, qu. 7, art. 1. also Summa Contra Gentiles, ch. 43; Locke, Essay on Human Understanding, bk. 2, ch. 16; Descartes, Principles of Philosophy, 1, 26; Spinoza, Ethics, I, Def. VI. Compare esp. the Spinoza passage with GA, p. 175.

36. GA, p. 175.

37. Ibid.

38. GA, p. 405.

39. Ibid.

40. Cantor writes in the Grundlagen that "it is always possible to bring any well-defined set into the form of a well-ordered set . . . ." This is "a law of thought which seems to me to be fundamental and rich in consequences. . . ." (GA, p. 169.)

41. Strictly speaking, an Anzahl is the order type of a well-ordered set; all simply ordered sets have order types, but only well-ordered sets have Anzahlen. Cf. GA, pp. 380, 420.

42. GA, p. 168.


44. By "number" here is of course meant 'ordinal number' since every set has a unique cardinal regardless of how it is ordered. In the Grundlagen however Cantor did not yet call cardinal numbers "numbers" but "powers" (Mächtigkeiten), and, at this stage, seemed to hold ordinality to be the primary concept. For the latter point, cf. GA, pp. 167, 420.

45. GA, p. 176. My emphasis.

46. Ibid. Cantor thus refutes Thomas's argument that God couldn't have created an actually infinite multitude (cf. above, n31) as follows. Insofar as God had the entire sequence of finite and transfinite numbers under his purview, and not simply the finite numbers, it was possible for him to comprehend an infinite multitude under a definite number and therefore to have a definite intention to create it. "... [T]he transfinite species [of number] stood with respect to the intentions of the Creator and his absolutely immeasurable Will (Willskraft) just as available to his commands as the finite numbers." (GA, p. 404, fn.)

47. Cantor writes: "... it will no longer be possible to deny the existence of the infinite while maintaining that of the finite on the basis of this difference [i.e., that counting procedures apply only to finite sets (Cantor took the concept of a well-ordering to be a generalization of the concept of counting)]; if the one is dropped, the other must be abandoned as well." (GA, p. 174.)

48. GA, p. 419.

49. GA, p. 205, fn. 2.


51. GA, p. 391.

52. Dauben appears to be on the right track in this matter when he mentions Cantor's views on the mathematical indeterminability of the absolute, and that Cantor
saw the ordinal number sequence $S$ as an “appropriate symbol” of it. But then he writes that Cantor “apparently... had never considered the formal, logical contradictions a system like the collection $[S]$ of all types engendered. When he first thought about such matters in 1895, he recognized that as a well-ordered set, $[S]$ should have a corresponding order type...,” from which the BF paradox follows. Thus, Dauben continues, “there was something inherently illegitimate in regarding $[S]$ as a set, as a consistent collection...” (Dauben, GC, p. 243.) If I am correct, it is false that Cantor ever (after 1883 anyway) thought that $S$ should have a corresponding order type. If one rejects this assumption, which is just an instance of BF2, as I’ve argued Cantor did, then $S$ engenders no “formal, logical contradictions” whatever.

53. GA, p. 441. The translation here is that found in van Heijenoort, From Frege to Gödel, p. 114.

54. The importance of the continuum problem for Cantor is clearly depicted throughout the course of GC.


56. GA, p. 441; van Heijenoort, From Frege to Gödel, p. 116, only I translate ‘ein fertiges Ding’ as ‘a completed thing’ instead of ‘one finished thing’.

57. Cf. the first sentences of “Infinite Sets and Plato’s Meikia” above.

58. GA, p. 205, n2; quoted above.


60. GA, p. 205, fn.


64. GA, p. 119.

65. GA, pp. 375, 404–05.

66. GA, pp. 167, 205, n2.


68. GA, p. 419.


70. GA, p. 280.

71. Ibid. p. 280.


73. GA, p. 352, n9.