8.2: Five Equivalence Rules

Comment: Recall that two statements are logically equivalent just in case they are true on exactly the same truth value assignments. Logically equivalent statements thus express the same information. Hence, given a statement $p$, one can always validly infer a logically equivalent statement $q$. This warrants the formation of rules of inference — known as equivalence rules — based upon logical equivalence. They provide us with explicit patterns of logical equivalence that we can use to infer new statements from given statements in a proof.

Because logically equivalent statements express the same information, one can always replace any statement-part $p$ of any statement $q$ with a statement $p'$ that is logically equivalent to $p$ and the resulting statement $q'$ will be logically equivalent to $q$; it will express the same information. Consequently, equivalence rules apply not only to the entire statement in a line of a proof, but to statement-parts of a statement in a line of a proof. This is the main difference between implicational rules and equivalence rules.

All of our equivalence rules are of the form $p :: q$, where the four-dot symbol ($::$) indicates that $p$ is logically equivalent to $q$. Such a rule tells us that, in the context of a proof, we may replace any occurrence of $p$ in a line of a proof (even if $p$ is only part of a statement on a line) with $q$ (or any occurrence of $q$ with $p$) and validly infer the result.

We illustrate with the first of our equivalence rules:

Rule 9: Double negation (DN)  

$p :: \sim \sim p$

Rule 9 thus tells us that if $p$ is a statement in a proof, or a part of a statement in a proof, we can replace it with $\sim \sim p$ and add the resulting statement to the proof as a further step, justified by DN. We can also go from “right to left”: that is, if $\sim \sim p$ is a (part of a) statement in a proof, we can replace it with $p$ and add the resulting statement to the proof as a further step, justified by DN.
**Example:** A proof for $\sim F \rightarrow \sim R$, $R : : F$

1. $\sim F \rightarrow \sim R$
2. $R : : F$

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**Rule 10: Commutation (Com)**

\[
p \lor q : : q \lor p \\
p \land q : : q \land p
\]

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**Example:** A proof for $P \rightarrow \sim (B \land O)$, $O \land B : : \sim P$

1. $P \rightarrow \sim (B \land O)$
2. $O \land B : : \sim P$
Rule 11: Association (As) \[ p \lor (q \lor r) :: (p \lor q) \lor r \]
\[ p \bullet (q \bullet r) :: (p \bullet q) \bullet r \]

Example: A proof for \((C \lor R) \lor D, \sim (R \lor D) \therefore C\)

1. \((C \lor R) \lor D\)
2. \(\sim (R \lor D) \therefore C\)

Rule 12: DeMorgan’s laws (DeM)
\[ \sim (p \lor q) :: \sim p \bullet \sim q \]
\[ \sim (p \bullet q) :: \sim p \lor \sim q \]

Example: A proof for \((E \bullet D) \lor (\sim E \bullet \sim D), \sim E \therefore \sim D\)

1. \((E \bullet D) \lor (\sim E \bullet \sim D)\)
2. \(\sim E \therefore \sim D\)
Rule 13: Contraposition (Cont) \[ p \rightarrow q : \sim q \rightarrow \sim p \]

Example: A proof for \((W \rightarrow D) \rightarrow \sim C, \sim D \rightarrow \sim W \therefore \sim C\)

1. \((W \rightarrow D) \rightarrow \sim C\)
2. \(\sim D \rightarrow \sim W \therefore \sim C\)

Tip 6: It is often useful to consider logically equivalent forms of the conclusion.

Example

1. \(\sim G \rightarrow \sim A\)
2. \(\sim H \rightarrow \sim B\)
3. \(\sim (G \cdot H) \therefore \sim (A \cdot B)\)
4.

\(\sim (A \cdot B)\)
Tip 7: Both conjunction and disjunction can lead to useful applications of De Morgan’s laws.

That is, when you see a conjunction or disjunction, or a negated conjunction or a negated disjunction, scan your premises and derived lines to see if a transformation using DeM might be useful.

Example

1. \( \sim J \lor \sim L \)
2. \( \sim (J \bullet L) \rightarrow \sim M \)
3. \( \sim E \lor (M \lor \sim S) \therefore \sim (S \bullet E) \)

\( \sim (S \bullet E) \)